

# CUT AND SEMI-CONJUGATE

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ABSTRACT. We define a very general class of rational functions  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  such that for every function  $f$  of this class, there exists a countable family of smooth curves  $\gamma_i$  and a critically finite function  $R$  such that the dynamical system obtained from  $f$  by cutting along the curves  $\gamma_i$  is topologically semi-conjugate to  $R$ .

We will consider topological dynamical systems on the 2-dimensional sphere  $S^2$  given by continuous maps  $f : S^2 \rightarrow S^2$ . If such a map  $f$  is fixed, it can be thought of as a geometric structure on the sphere. Namely, we can imagine that the sphere is equipped with arrows connecting  $x$  with  $f(x)$  for all points  $x \in S^2$ . Of course, the shape of the arrows does not matter, the only essential information being which pairs of points are connected by arrows.

Homeomorphisms of spheres with arrows (i.e. topological conjugacies) do not change topological dynamics. Thus, to change it, one needs some sort of topological surgery including discontinuous operations such as cuts. The simplest way to cut the sphere is to make a cut along some simple curve. However, if the sphere comes with arrows depicting a dynamical system  $f$ , then a cut along a curve creates some problems. Namely, if we cut through the tip of an arrow, then the arrow splits. This is problematic because a continuous map cannot create two different arrows beginning at the same point. To resolve this problem, one needs to make additional cuts, namely, through the beginnings of all the arrows, whose ends are in the cut. In other words, if we cut the sphere along a curve, then we also need to cut it along the pre-image of this curve. This creates further problems as we cut through the tips of some arrows again. This, if we cut along a simple curve  $Z \subset S^2$ , then we also need to cut along  $f^{-1}(Z)$ , along  $f^{-2}(Z)$ , and so on. This leads to countably many cuts. As we make all these cuts, we obtain some topological space  $X$  called a *sphere with cuts*.

The sphere with cuts  $X$  carries well-defined arrows. In other terms, we have a well-defined continuous map  $F : X \rightarrow X$ . Note that the topological dynamical system  $F : X \rightarrow X$  is uniquely determined by the map  $f$  and the first curve  $Z$ , along which we cut (we call this curve the *initial cut*). We will also consider a more

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general set-up, in which  $Z$  is a union of several simple curves. We will refer to the map  $F$  as the *map obtained from  $f$  by cutting*.

More precisely, the set  $X$  and the map  $F$  are defined as follows. Let  $U_n$  be the complement to  $Z \cup f(Z) \cup \dots \cup f^{\circ n}(Z)$  in  $S^2$ . Define the compact space  $X_n$  as the Caratheodory compactification of  $U_n$  (i.e. the union of  $U_n$  and the set of all prime ends of  $U_n$  equipped with a suitable topology). The inclusion  $U_{n+1} \rightarrow U_n$  gives rise to a continuous map  $\iota_n : X_{n+1} \rightarrow X_n$ . We define the topological space  $X$  as the inverse limit of the system of spaces  $X_n$  and maps  $\iota_n$ . The maps  $F_n : X_{n+1} \rightarrow X_n$  corresponding to  $f : U_{n+1} \rightarrow U_n$  define the continuous map  $F : X \rightarrow X$ .

After countably many cuts, the orbits of almost all points remain unchanged. Thus we can think that the dynamics of the map  $F : X \rightarrow X$  is not too much different from the dynamics of the map  $f$ . On the other hand, as a topological space,  $X$  can be very different from the sphere, e.g. it can have uncountably many connected components. In a very general situation described below, there is a semi-conjugacy between the topological dynamical system  $F : X \rightarrow X$  and some *hyperbolic critically finite* (i.e. very good!) function  $R$ . Recall that a function  $R$  is hyperbolic and critically finite if each of its critical points gets eventually mapped to a periodic critical point. Thus we can study the dynamics of  $F : X \rightarrow X$  (hence also the dynamics of  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ ) using a semi-conjugacy with a hyperbolic rational function. In fact, the good rational function  $R$  can be usually chosen in infinitely many ways.

To state a precise theorem, we will need a relation on the set of all rational functions. It resembles combinatorial equivalence but is in fact much weaker. Let  $f$  and  $R$  be rational functions, and suppose that  $R$  has a finite post-critical set  $P_R$ . We say that the function  $R$  *can represent* the function  $f$ , if  $R$  is homotopic to a map topologically conjugate to  $f$  through branched coverings so that the homotopy preserves the finite set  $R(P_R)$ . The difference with the usual definition of combinatorial equivalence is only that the set  $P_R$  is replaced with the set  $R(P_R)$ . But, in contrast to combinatorial equivalence, there are usually many (even infinitely many) critically finite rational functions  $R$  that can represent the same function  $f$ . However, if  $R$  can represent  $f$ , then these two functions must have the same structure of super-attracting orbits. On the other hand, the function  $f$  can have many critical points, whose orbits are infinite and show as complicated behavior as they please. We require that  $R$  be hyperbolic. In particular, it must have at least one super-attracting cycle. This means that  $f$  must also have at least one super-attracting cycle; moreover, the structure of super-attracting cycles for  $f$  must be the same as for  $R$ . This is almost all we need from  $f$ .

We now state the main result.

**Theorem 1.** *Let  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  be a rational function, and  $R : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  a critically finite hyperbolic function that can represent  $f$ . Then there exist an initial cut  $Z$  (which is a finite union of smooth simple curves), the corresponding sphere with cuts  $X$  and the map  $F : X \rightarrow X$  obtained from  $f$  by cutting, such that the*

*topological dynamical system  $F : X \rightarrow X$  is semi-conjugate to the dynamical system  $R : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ .*

Recall that the dynamical system  $F : X \rightarrow X$  is determined by the choice of the initial cut  $Z$ . This choice can be made very explicit. Moreover, the set  $Z$  is defined only up to a suitable homotopy, so that we can arrange that all curves that appear in  $Z$  are analytic, or even real semi-algebraic (alternatively, we can make them broken lines).

Below, we give some examples, where the choice of  $Z$  is made more explicit. To this end, we consider certain complex one-dimensional parameter spaces of quadratic rational functions.

It is a general observation and philosophy that the dynamical behavior of a rational function is determined by the behavior of its critical orbits. A rational function of degree two has two critical points. Thus, if we fix a simple dynamical behavior of one of these points, then we are left with only one “free” critical point. This makes the corresponding parameter space complex one-dimensional. To fix a simple behavior of a critical point, we can e.g. require that this point be periodic of period  $k$ . The space of such rational functions is  $Per_k(0)$  (in this notation, 0 stands for the multiplier of a  $k$ -periodic point). More precisely,  $Per_k(0)$  consists of Möbius conjugacy classes of rational functions  $R$  of degree 2 with marked critical points  $c_1, c_2$  such that  $R^{ok}(c_1) = c_1$ .

The parameter spaces  $Per_k(0)$  are one-dimensional slices of the parameter space of all degree 2 rational functions (with marked critical points). E.g., for  $k = 1$ , we obtain the space of quadratic polynomials  $z^2 + c$ . Indeed, if a rational function has a fixed critical point, then, mapping this point to infinity by a suitable Möbius conjugacy, we can make this function into a quadratic polynomial; the term with  $z$  can be killed by a parallel translation.

The set of hyperbolic functions in  $Per_k(0)$  is an open set, whose components are called *hyperbolic components*. Recall that a hyperbolic function  $R$  with periodic critical point  $c_1$  (and the corresponding hyperbolic component in  $Per_k(0)$ ) is said to be of *type C* (C stands for “capture”; not to be confused with the capture operation of B. Wittner) if  $c_2$  lies in the basin of the cycle  $c_1, \dots, R^{ok-1}(c_1)$ , but not in the immediate basin. Recall also that any hyperbolic component of type C in  $Per_k(0)$  contains a unique critically finite Möbius conjugacy class of rational functions. In  $Per_k(0)$ , *any hyperbolic critically finite function of type C can represent any non-hyperbolic function.*

So let  $f$  be any non-hyperbolic function and  $R$  any critically finite type C hyperbolic function, whose classes are in  $Per_k(0)$ . *The initial cut  $Z$  is a simple curve containing the non-periodic critical point of  $f$ .* It is defined as the full pre-image of some simple curve  $\beta$  under  $f$ , so that the map  $f|_Z$  is two-to-one except that the critical point  $c_2$  is the only preimage of the critical value  $v = f(c_2)$ . The curve  $\beta$  connects the non-periodic critical value  $v$  of  $f$  with a certain pre-periodic point  $w$  that gets eventually mapped to the periodic critical point of  $f$ . *The curve  $\beta$  is*

only defined up to a homotopy relative to the forward orbit of  $w$ . The following is a useful property that characterizes the homotopy class of  $\beta$ . Let  $\sigma_\beta$  be a self-homeomorphism of the Riemann sphere that is equal to the identity outside a small neighborhood of  $\beta$  and that takes  $v$  to  $w$ . Then  $\sigma_\beta \circ f$  is a critically finite branched covering, whose combinatorial equivalence class is well defined. The curve  $\beta$  should be chosen so that to make the covering  $\sigma_\beta \circ f$  combinatorially equivalent to  $R$ . There is a simple and rather explicit construction of M. Rees [2] that produces such a curve  $\beta$ .

Suppose that the class of  $f$  lies on the boundary of the type C hyperbolic component containing the class of  $R$ . Then  $\beta$  can be chosen as the closure of an internal ray in a Fatou component of  $f$ , whose boundary contains the critical value  $v$ . In this case, all cuts we need to make in the sphere are disjoint simple curves. To obtain the topological dynamics of  $R$  from the topological dynamics of  $f$ , we need to cut along all pullbacks of  $Z$  and then reglue them differently (this is similar to cutting the plane along the interval  $[-1, 1]$ , opening up the cut to form the circle  $x^2 + y^2 = 1$ , so that the previously identified points have equal  $x$ -coordinates, and gluing the circle so that the points with equal  $y$ -coordinates are identified). This topological surgery is called *regluing*; it was introduced in [3]. Regluing is reversible: if  $R$  can be obtained from  $f$  by regluing, then also  $f$  can be obtained from  $R$  by regluing. This provides topological models for non-hyperbolic rational functions, corresponding to boundary points of type C hyperbolic components in  $Per_k(0)$ .

A proof of Theorem 1 is given in preprint [4]. The proof is rather technical; it uses a version of Thurston's algorithm [1]. Thurston's algorithm is applied to the composition of  $F : X \rightarrow X$  with the projection  $X \rightarrow \mathbb{CP}^1$  rather than to a branched covering of the sphere.

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